

GRADED BETTI NUMBERS OF I -GOOD FILTRATIONS

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ABSTRACT. In this work we generalize our previous results about graded Betti numbers of powers of homogenous ideals in \mathbb{Z} -graded algebra over Noetherian local ring and finally we investigate the structure of Tor module of I -good filtrations.

1. INTRODUCTION

Non-standard Hilbert functions first raised in Gabber's proof of Serre non-negativity conjecture [14]. It has been studied by several authors [8, 14, 15]. As it was noticed in [1], the module $\oplus_t \text{Tor}_i^S(I^t, k)$ for a homogeneous ideal I in graded ring S has the structure of a finitely generated graded module over a non-standard graded polynomial ring over k , from which one can deduce the behavior of $\text{Tor}_i^S(I^t, k)$ when t varies.

This graded structure was the core of our previous work about quasi-polynomial behavior of the Betti numbers of homogeneous ideals in a polynomial ring over a field, using the concept of vector partition functions. In this work, we generalize the behavior to the case of homogeneous ideals in G -graded algebra over Noetherian local ring.

In order to be able to consider integral closures of powers, and other constructs, we use the notion of I -good filtrations. This is a series $\mathcal{J} = \{\mathcal{J}_n\}_{n \geq 0}$ of ideals such that $\oplus_{n \geq 0} \mathcal{J}_n$ is a finite module over the Rees ring.

Theorem 1.1. (See Theorem 4.2) Let $S = A[x_1, \dots, x_n]$ be a graded algebra over a Noetherian local ring $(A, m) \subset S_0$. Let $\mathcal{J} = \{\mathcal{J}_n\}_{n \geq 0}$ be an I -good filtration of ideals \mathcal{J}_n of S and $\mathcal{J}_1 = (f_1, f_2, \dots, f_r)$ with $\deg f_i = d_i$ be \mathbb{Z} -homogenous ideal in S , and let $R = S[T_1, \dots, T_r]$ be a bigraded polynomial extension of S with $\deg(T_i) = (d_i, 1)$ and $\deg(a) = (\deg(a), 0) \in \mathbb{Z} \times \{0\}$ for all $a \in S$. Then,

(1) all i, j :

$\text{Tor}_i^A(\text{Tor}_j^R(\mathcal{R}_{\mathcal{J}}, A), k)$ is finitely generated $k[T_1, \dots, T_r]$ -module.

(2) There exist, $t_0, m, D \in \mathbb{Z}$, linear functions $L_i(t) = a_i t + b_i$, for $i = 0, \dots, m$, with a_i among the degrees of the minimal generators of I and $b_i \in \mathbb{Z}$, and polynomials $Q_{i,j} \in \mathbb{Q}[x, y]$ for $i = 1, \dots, m$ and $j \in 1, \dots, D$, such that, for $t \geq t_0$,

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- (i) $L_i(t) < L_j(t) \Leftrightarrow i < j$,
- (ii) If $\mu < L_0(t)$ or $\mu > L_m(t)$, then $\text{Tor}_i^S(\mathcal{J}_t, k)_\mu = 0$.
- (iii) If $L_{i-1}(t) \leq \mu \leq L_i(t)$ and $a_i t - \mu \equiv j \pmod{(D)}$, then
$$\dim_k \text{Tor}_i^S(\mathcal{J}_t, k)_\mu = Q_{i,j}(\mu, t).$$

2. PRELIMINARIES

In this section, we give brief recall on necessary notations and terminology used in the article.

Let $S = k[x_1, \dots, x_n]$ be a polynomial ring over a field k . Let G be an abelian group. A G -grading of S is a morphism $\deg : \mathbb{Z}^n \rightarrow G$ and If G is torsion-free and $S_0 = k$, the grading is positive. Criteria of positivity are given in [10, 8.6]. When $G = \mathbb{Z}^d$ and the grading is positive, (generalized) Laurent series are associated to finitely generated graded modules:

Definition 2.1. The Hilbert function of a finitely generated module M over a positively graded polynomial ring is the map:

$$\begin{aligned} HF(M; -) : \mathbb{Z}^d &\longrightarrow \mathbb{N} \\ \mu &\longmapsto \dim_k(M_\mu). \end{aligned}$$

The Hilbert series of M is the Laurent series

$$H(M; t) = \sum_{\mu \in \mathbb{Z}^d} \dim_k(M_\mu) t^\mu.$$

Let M be a finitely generated \mathbb{Z}^d -graded S -module. It admits a finite minimal graded free S -resolution

$$\mathbb{F}_\bullet : 0 \rightarrow F_u \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0.$$

Writing

$$F_i = \bigoplus_\mu S(-\mu)^{\beta_{i,\mu}(M)},$$

the minimality shows that $\beta_{i,\mu}(M) = \dim_k(\text{Tor}_i^S(M, k))_\mu$, as the maps of $\mathbb{F}_\bullet \otimes_S k$ are zero. We also recall that the support of a \mathbb{Z}^d -graded module N is

$$\text{Supp}_{\mathbb{Z}^d}(N) := \{\mu \in \mathbb{Z}^d \mid N_\mu \neq 0\}.$$

2.1. Partition function. Let e_i be the standard basis of the space \mathbb{R}^r for $1 \leq i \leq r$. Let f be a linear map $f : \mathbb{R}^r \rightarrow \mathbb{R}^d$ defined by $f(e_i) = v_i$ and denote by V the linear span of $[v_1, \dots, v_r]$. For $a \in V$ consider the following convex polytope:

$$P(a) := f^{-1}(a) \cap \mathbb{R}_{\geq 0}^r = \{x = (x_1, \dots, x_r) \in \mathbb{R}^r \mid \sum_{i=1}^r x_i v_i = a; x \geq 0\}.$$

Definition 2.2. The function $\varphi : \mathbb{N}^d \rightarrow \mathbb{N}$ defined by $\varphi_A(a) = \sharp(f^{-1}(a) \cap \mathbb{Z}_{\geq 0}^r)$ is called vector partition function corresponding to the matrix $A = (v_1, \dots, v_r)$.

For more details about the vector partition functions in particular the definition of chambers, chamber complex and quasipolynomials we use the terminology of [3, 5, 16].

Now, we recall the vector partition function theorem:

Theorem 2.3. (See [16, Theorem 1]) *For each chamber C of maximal dimension in the chamber complex of A , there exist a polynomial P of degree $n - d$, a collection of polynomials Q_σ and functions $\Omega_\sigma : G_\sigma \setminus \{0\} \rightarrow \mathbb{Q}$ indexed by non-trivial $\sigma \in \Delta(C)$ such that, if $u \in \mathbb{N}A \cap \overline{C}$,*

$$\varphi_A(u) = P(u) + \sum \{\Omega_\sigma([u]_\sigma) \cdot Q_\sigma(u) : \sigma \in \Delta(C), [u]_\sigma \neq 0\}$$

where $[u]_\sigma$ denotes the image of u in G_σ . Furthermore, $\deg(Q_\sigma) = \#\sigma - d$.

Corollary 2.4. [2] *For each chamber C of maximal dimension in the chamber complex of A , there exists a collection of polynomials Q_τ for $\tau \in \mathbb{Z}^d/\Lambda$ such that*

$$\varphi_A(u) = Q_\tau(u), \text{ if } u \in \mathbb{N}A \cap \overline{C} \text{ and } u \in \tau + \Lambda_C,$$

where $\Lambda_C = \cap_{\sigma \in \Delta(C)} \Lambda_\sigma$

Notice that setting Λ for the intersection of the lattices Λ_σ with σ maximal, the class of u mod Λ determines the class of u mod Λ_C , hence the corollary holds with Λ in place of Λ_C .

3. STRUCTURE OF TOR MODULE OF REES ALGEBRA

Let $S = A[x_1, \dots, x_n]$ be a graded algebra over a commutative noetherian local ring $S_0 = (A, m)$ with residue field k and set $R = S[T_1, \dots, T_r]$ and $B = k[T_1, \dots, T_r]$. We set $\deg(T_i) = (d_i, 1)$ and extended the grading from S to R by setting $\deg(x_i) = (\deg(x_i), 0)$. Let M be a finitely generated graded S -module and I be a graded S -ideal generated in degrees d_1, \dots, d_r . The natural anto map $R \rightarrow \mathcal{R}_I := \bigoplus_{t \geq 0} I^t$ sending T_i to f_i makes $M\mathcal{R}_I = \bigoplus_{t \geq 0} MI^t$ a finitely generated graded R -module. In this section we use the two important following fact that were already at the center of the work [1]. The first one is that $\text{Tor}_i^R(M\mathcal{R}_I, B)$ is a finitely generated graded B -module. The second is that :

$$\text{Tor}_i^R(M\mathcal{R}_I, B)_{(*,t)} = \text{Tor}_i^S(MI^t, k).$$

In particular, it provides a B -structure on $\bigoplus_t \text{Tor}_i^S(MI^t, k)$ making it a finitely generated B -module. Slightly more generally, it was shown in [1] that the following holds.

Theorem 3.1. [1] *Let $S = A[x_1, \dots, x_n]$ be a \mathbb{G} -graded algebra over Noetherian local ring (A, m, k) . Let $I = (f_1, f_2, \dots, f_r)$ with $\deg f_i = d_i$ be \mathbb{G} -homogenous ideal in S , and let $R = S[T_1, \dots, T_n]$ be a bigraded polynomial extension of S with $\deg(T_i) = (d_i, 1)$ and $\deg(a) = (\deg(a), 0) \in \mathbb{G} \times \{0\}$ for all $a \in S$. Let M be a finitely generated \mathbb{G} -graded S -module. Then for all i, j :*

(1) $\text{Tor}_i^A(\text{Tor}_j^R(M\mathcal{R}_I, A), k)$ is finitely generated $k[T_1, \dots, T_r]$ -module .

(2) $\text{Tor}_i^R(M\mathcal{R}_I, k)$ is finitely generated $k[T_1, \dots, T_r]$ -module .

Theorem 3.2. *In the above situation assume that I is a homogeneous ideal in S and $\mathbb{G} = \mathbb{Z}$.*

Then there exists, $t_0, m, D \in \mathbb{Z}$, linear functions $L_i(t) = a_i t + b_i$, for $i = 0, \dots, m$, with a_i among the degrees of the minimal generators of I and $b_i \in \mathbb{Z}$, and polynomials $Q_{i,j} \in \mathbb{Q}[x, y]$ for $i = 1, \dots, m$ and $j \in 1, \dots, D$, such that, for $t \geq t_0$,

(i) $L_i(t) < L_j(t) \Leftrightarrow i < j$,

(ii) If $\mu < L_0(t)$ or $\mu > L_m(t)$, then $\text{Tor}_i^S(I^t, k)_\mu = 0$.

(iii) If $L_{i-1}(t) \leq \mu \leq L_i(t)$ and $a_i t - \mu \equiv j \pmod{D}$, then

$$\dim_k \text{Tor}_i^S(I^t, k)_\mu = Q_{i,j}(\mu, t).$$

Proof. By the above the theorem we know that $\text{Tor}_i^R(M\mathcal{R}_I, k)$ is finitely generated $k[T_1, \dots, T_r]$ -module then the result follows from [2, Proposition 4.5]. □

4. STRUCTURE OF TOR MODULE OF HILBERT FILTRATIONS

To study blowup algebras, Northcott and Rees defined the notion of reduction of an ideal I in a commutative ring R . An ideal $J \subseteq I$ is a reduction of I if there exists r such that $J I^r = I^{r+1}$ (equivalently this hold for $r \gg 0$). An important fact about reduction of ideals is that this property is equivalent to the fact that

$$\mathcal{R}_J = \bigoplus_n J^n \rightarrow \mathcal{R}_I = \bigoplus_n I^n$$

is a finite morphism. Okon and Ratliff in [11] extended the above notion of reduction to the case of filtrations by setting the following definition :

Definition 4.1. Let R be a ring, I an R -ideal and $\mathcal{J} = \{\mathcal{J}_n\}_{n \geq 0}$ and $\mathcal{I} = \{\mathcal{I}_n\}_{n \geq 0}$ two filtration on R :

(1) $\mathcal{J} \leq \mathcal{I}$ if $\mathcal{J}_n \subseteq \mathcal{I}_n$ for all $n \geq 0$.

(2) \mathcal{J} is a reduction of \mathcal{I} if $\mathcal{J} \leq \mathcal{I}$ and there exists a positive integer d such that $\mathcal{I}_n = \sum_{i=0}^d \mathcal{J}_{n-i} \mathcal{I}_i$ for all $n \geq 1$.

(3) \mathcal{J} is a I -good filtration if $I\mathcal{J}_i \subseteq \mathcal{J}_{i+1}$ for all $i \geq 0$ and $\mathcal{J}_{n+1} = I\mathcal{J}_n$ for all $n \gg 0$.

Opposite to the ideal case, minimal reductions of a filtration does not exist in general. But Hoa and Zarzuela showed in [9] the existence of a minimal reduction for I -good filtrations.

If $\mathcal{J} = \{\mathcal{J}_n\}_{n \geq 0}$ is an I -good filtration on R , then $\mathcal{R}_{\mathcal{J}} := \bigoplus_{n \geq 0} \mathcal{J}_n$ is a finite \mathcal{R}_I -module [4, Theorem III.3.1.1]. This is why we are interested about I -good filtration to generalize the previous results. The following theorem explain the structure of Tor module of I -good filtrations :

Theorem 4.2. *Let $S = A[x_1, \dots, x_n]$ be a graded algebra over a Noetherian local ring $(A, m, k) \subset S_0$. Let $\mathcal{J} = \{\mathcal{J}_n\}_{n \geq 0}$ be an I -good filtration of \mathbb{Z} -homogeneous ideals in S , and $\mathcal{J}_1 = (f_1, f_2, \dots, f_r)$ with $\deg f_i = d_i$. Let $R = S[T_1, \dots, T_n]$ be a bigraded polynomial extension of S with $\deg(T_i) = (d_i, 1)$ and $\deg(a) = (\deg(a), 0) \in \mathbb{Z} \times \{0\}$ for all $a \in S$.*

(1) *Then for all i :*

$\text{Tor}_i^R(\mathcal{R}_{\mathcal{J}}, k)$ *is a finitely generated $k[T_1, \dots, T_r]$ -module .*

(2) *There exist, $t_0, m, D \in \mathbb{Z}$, linear functions $L_i(t) = a_i t + b_i$, for $i = 0, \dots, m$, with a_i among the degrees of the minimal generators of I and $b_i \in \mathbb{Z}$, and polynomials $Q_{i,j} \in \mathbb{Q}[x, y]$ for $i = 1, \dots, m$ and $j \in 1, \dots, D$, such that, for $t \geq t_0$,*

(i) $L_i(t) < L_j(t) \Leftrightarrow i < j$,

(ii) *If $\mu < L_0(t)$ or $\mu > L_m(t)$, then $\text{Tor}_i^S(\varphi(t), k)_{\mu} = 0$.*

(iii) *If $L_{i-1}(t) \leq \mu \leq L_i(t)$ and $a_i t - \mu \equiv j \pmod{D}$, then*

$$\dim_k \text{Tor}_i^S(\mathcal{J}_t, k)_{\mu} = Q_{i,j}(\mu, t).$$

Proof. Recall that $\mathcal{R}_{\mathcal{J}}$ is a finite \mathcal{R}_I -module, hence a finitely generated \mathbb{Z}^2 -graded R -module. Let F_{\bullet} be a $\mathbb{Z} \times \mathbb{Z}$ -graded minimal free resolution of \mathcal{R}_{φ} over R . Each F_i is of finite rank due to the Noetherianity of A . The graded stand $F_{\bullet}^t := (F_{\bullet})_{*,t}$ is a \mathbb{Z} -graded free resolution of \mathcal{J}_t over $S = R_{(*,0)}$. Thus,

$$\text{Tor}_i^S(\mathcal{J}_t, k) = H_i(F_{\bullet}^t \otimes_S k).$$

Moreover, taking homology respects the graded structure and therefore,

$$H_i(F_{\bullet}^t \otimes_S k) = H_i(F_{\bullet} \otimes_R R/m + nR)_{(*,t)},$$

where $\mathbf{n} = (x_1, \dots, x_n)$ is the homogeneous irrelevant ideal of S . It follows that $\text{Tor}_j^R(\mathcal{R}_I, k)$ is a finitely generated graded $k[T_1, \dots, T_r]$ -module. The second fact then follow from [2, Proposition 4.5].

□

This in particular applies to the following situations:

- I is a graded ideal of S and S is an analytically unramified ring without nilpotent elements. Then the integral closure filtration $\mathcal{J} = \{\overline{I}^n\}$ is I -good filtration[13].
- I is a graded ideal of S , then the rattliff-Rush closure filtration $\mathcal{J} = \{\widetilde{I}^n\}$ is an I -good filtration[12].

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REFERENCES

- [1] A. Bagheri, M. Chardin and H.T. Hà. *The eventual shape of the Betti tables of powers of ideals*. To appear in Math. Research Letters.
- [2] Bagheri, Amir, and Kamran Lamei. Graded Betti numbers of powers of ideals. Preprint. arXiv:1308.0943.
- [3] S. Blakley. Combinatorial remarks on partitions of a multipartite number. Duke Math. J. **31**(1964), 335-340.
- [4] N. Bourbaki. Commutative algebra, Addison Wesley, Reading, 1972.
- [5] M. Brion and M. Vergne. Residue formulae, vector partition functions and lattice points in rational polytopes. J. Amer. Math. Soc.. **10**(1997), 797-833.
- [6] W. Bruns and J. Herzog. Cohen-Macaulay rings. Cambridge Studies in Advanced Mathematics, **39**. Cambridge University Press, Cambridge, 1993.
- [7] De Loera, J. A., Hemmecke, R., Tauzer, J., and Yoshida, R. (2004). Effective lattice point counting in rational convex polytopes. Journal of symbolic computation, 38(4), 1273-1302.
- [8] Hoang, N. D., and Trung, N. V. (2003). Hilbert polynomials of non-standard bigraded algebras. Mathematische Zeitschrift, 245(2), 309-334.
- [9] Hoa, L Tun, and Santiago Zarzuela. Reduction number and a-invariant of good filtrations. Communications in Algebra 22.14 (1994): 5635-5656.
- [10] E. Miller and B. Sturmfels. Combinatorial commutative algebra. Graduate Texts in Mathematics, 227. Springer-Verlag, New York, 2005
- [11] Okon, J. S., and Louis Ratliff. Reductions of filtrations. Pacific Journal of Mathematics 144.1 (1990): 137-154.
- [12] L. J. Ratliff Jr. and D. E. Rush. Two notes on reduction of ideals. Indiana Univ. Math. J., 27, 929-934, 1978.
- [13] D. Rees. A note on analytically unramified local rings. J. London Math. Soc. 36, 24-28, 1961.
- [14] Roberts, P. C. (1998). Recent developments on Serre's multiplicity conjectures: Gabber's proof of the nonnegativity conjecture. ENSEIGNEMENT MATHEMATIQUE, 44, 305-324.
- [15] Roberts, P. (2000). Intersection multiplicities and Hilbert polynomials. Michigan Math. J, 48, 517-530.
- [16] B. Sturmfels. On vector partition functions. J. Combinatorial Theory, Series A **72**(1995), 302-309.
- [17] G. Whieldon. Stabilization of Betti tables. Preprint. arXiv:1106.2355.

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